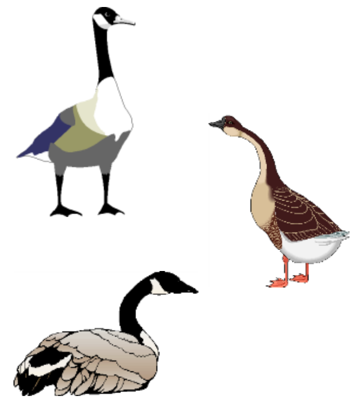
CH 2 – SETS

A *bunch* of bananas, a *gaggle* of geese — each is an example of a ***set***, referred to as an *ensemble* in French. The idea of a set is so fundamental that it doesn't even have a definition, yet sets form the foundation of all mathematics.



❑ BASIC SET DEFINITIONS

A ***set*** is a well-defined collection of objects (but what's a collection?). Well-defined means that, for instance, "all whole numbers greater than 10" is a set, but "all tall people" is not a set (it's too vague).

Consider the sets

$$A = \{1, 3, 5\} \qquad B = \{1, 3, 5, 7, 10\}$$

The ***elements*** (or members) of set A are 1, 3, and 5, surrounded by (curly) braces. We say "3 is an element of A" and write "$\mathbf{3 \in A}$". We can also say that "12 is not an element of A", in which case we write "$\mathbf{12 \notin A}$".

In the sets A and B above, notice that all the elements of A are also elements of B. In a sense, A is contained in B. We say that "A is a ***subset*** of B" and write "$\mathbf{A \subseteq B}$". Can you see that B is not a subset of A? We write "$B \not\subseteq A$".

There's a very special set that contains *no* elements. It is called the ***null set*** (or the empty set), and is written $\varnothing$. Some teachers prefer to write { } for the null set.

Most books use the symbol $\mathbb{N}$ to denote the set of all **natural numbers**:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Also, we'll use the symbol $\mathbb{Z}$ to represent the set of ***integers***:

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Z is the first letter of ***Z***ahl, the German word for *number*.

Homework

1. Consider the collection of all students registered in this class. Is this a set? Why or why not?

Do you see that $\mathbb{N} \subseteq \mathbb{Z}$?

2. Consider the collection of all students at the college with a high GPA. Is this a set? Why or why not?

3. Fill in the blanks:
   a. 7 ____ { all primes } b. 13 ____ { all even numbers }

4. Fill in the blanks:
   a. 6 ____ {3, 6, 9} b. {6} ____ {3, 6, 9}

5. a. T/F: {students in this class} $\subseteq$ { DVC students}
   b. T/F: {people in this class} $\subseteq$ { DVC students}

6. T/F: {primes} $\subseteq$ {odd natural numbers}
   Hint: The first few primes are 2, 3, 5, 7, 11, 13, ...

7. Explain why $\{7, 8, 9\} \not\subseteq \{7, 9, 11, 13, 20\}$.

8. T/F: $\{\emptyset\} = \emptyset$. Be sure to explain your answer.

9. Which statement is true, $\{1, 9, 21\} \subseteq \mathbb{N}$ or $\mathbb{N} \subseteq \{1, 9, 21\}$?

10. Prove that $\mathbb{N} \subseteq \mathbb{Z}$. Now prove that $\mathbb{Z} \not\subseteq \mathbb{N}$.

11. Fill in the blank: If $A \subseteq B$ and $B \subseteq A$, then ________.

12. T/F: $\{1, 3\} \in \{\{a\}, \{x, y\}, \{1, 3\}\}$

❑ *Important Properties of Sets*

- The elements of a set may be explicitly listed, such as the set of positive even numbers: {2, 4, 6, 8, . . .}. The positive even numbers can also be written $\{n \in \mathbb{N} \mid n \text{ is even}\}$, where the vertical bar, $\mid$, is read *"such that."* Some books write this with a colon instead of a vertical bar: $\{n \in \mathbb{N}: n \text{ is even}\}$

- The order in which the elements of a set are listed doesn't matter. For example, {1, 2, 3} = {2, 3, 1}.

- Repeated elements in a set can be thrown out without changing the set. For instance, {a, b, a, c} = {a, b, c}.

- Is every element of the set {1, 2, 3} also in the set {1, 2, 3}? Yes, so $\{1, 2, 3\} \subseteq \{1, 2, 3\}$. In general, for any set A,

$$\mathbf{A} \subseteq \mathbf{A}$$

- We now contend that $\varnothing \subseteq \{1, 2, 3\}$. This is really strange, but look at it this way: If the null set were <u>not</u> a subset of {1, 2, 3}, then there would have to be something in the null set that <u>fails</u> to be in {1, 2, 3}. But this is impossible since there's nothing in the null set! Thus, it can <u>not</u> be the case that the null set is <u>not</u> a subset of {1, 2, 3}, and therefore it <u>is</u> a subset: $\varnothing \subseteq \{1, 2, 3\}$. This may be hard to fathom, so you may have to accept it on faith. Just memorize the following fact: If A is <u>any</u> set,

$$\varnothing \subseteq \mathbf{A}$$

Homework

13. List the elements of $\{x \in \mathbb{N} \mid x \leq 5\}$.

14. List the elements of $\{x \in \mathbb{N} \mid 5 \leq x < 10\}$.

15. Write the set $\{2, 4, 6, 8, 10\}$ in the form $\{x \in \mathbb{N} \mid x \text{ is} \ldots\}$.

16. Write the set $\{2, 3, 5, 7, 11, 13\}$ in the form $\{x \in \mathbb{N} \mid x \text{ is} \ldots\}$.

17. T/F: $\{6, 7, 8, 7, 4\} = \{4, 7, 6, 8\}$. Explain.

18. T/F: $\{a, b, c, d, e\} = \{b, c, d, e\}$. Explain.

19. T/F: $\{1, 5, 9\} \subseteq \{1, 5, 9\}$

20. T/F: $\varnothing \subseteq \varnothing$. Explain.

21. The number of elements in a set is called its ***cardinality***. For example, the cardinality of $\{2, 4, 8, 16, 32\}$ is 5. What is the cardinality of $\varnothing$? What is the cardinality of $\{1, 3, 5, 7, \ldots, 21\}$?

22. Let A be a set with cardinality 100. What is the cardinality of $\{A\}$? [This is tricky.]

23. What is the cardinality of $\mathbb{N}$? What is the cardinality of the set of even numbers? Do $\mathbb{N}$ and the even numbers have the same cardinality?

❑ *COMBINING SETS*

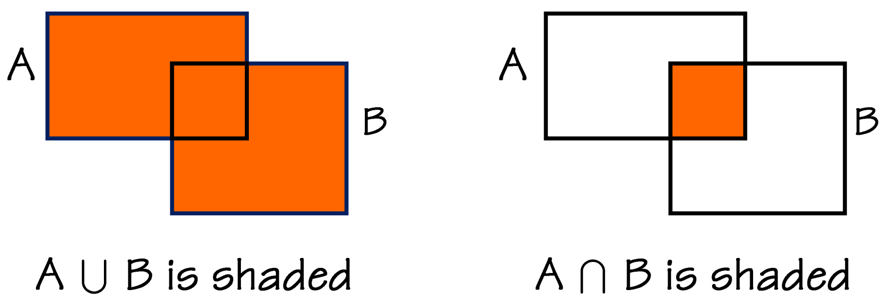
The ***union*** of two sets A and B is the new set obtained by lumping together all the elements that are in A <u>or</u> B (or both). The union of A and B is written $\mathbf{A} \cup \mathbf{B}$. For example,

$\{1, 2, 7\} \cup \{2, x, \Delta\} = \{1, 2, 7, x, \Delta\}$

The ***intersection*** of sets A and B is the new set obtained by extracting just the elements in both A <u>and</u> B — that is, the elements that A and B have *in common*. The intersection of A and B is written $\mathbf{A} \cap \mathbf{B}$. For example,

$\{a, b, R\} \cap \{b, c, R\} = \{b, R\}$

Union and intersection can be displayed as *Venn diagrams*:



The following are the official definitions of union and intersection:

$\mathrm{A} \cup \mathrm{B} = \{x \mid x \in \mathrm{A} \text{ or } x \in \mathrm{B}\}$

$\mathrm{A} \cap \mathrm{B} = \{x \mid x \in \mathrm{A} \text{ and } x \in \mathrm{B}\}$

<u>EXAMPLE 1:</u>

A. $\{a, b, c\} \cup \{d, w, z\} = \{a, b, c, d, w, z\}$

B. $\{1, 3, 5, 7\} \cap \{2, 4, 6\} = \varnothing$

C. $\{m, n, p\} \cup \{m, n, o, p, q\} = \{m, n, o, p, q\}$

D. $\{7, 8, 9\} \cap \{7, 8, 9, 10\} = \{7, 8, 9\}$

E. $\{a, 1, 2, \pi\} \cup \varnothing = \{a, 1, 2, \pi\}$

F. $\{x, y, z\} \cap \varnothing = \varnothing$

Homework

24. Sketch a Venn diagram representing the statement: $A \subseteq B$.

25. Use a Venn diagram to prove:

    If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

26. Find $\{a, b, c\} \cup \{b, d, w, z\} \cup \{a, d, w, m, z\}$.

27. Find $\{1, 3, 5\} \cap \{3, 5, 7\} \cap \{5, 7, 9\}$.

28. Two sets A and B are said to be ***disjoint*** if $A \cap B = \varnothing$; for example, the set of even numbers and the set of odd numbers are disjoint. Sketch a Venn diagram of two disjoint sets.

29. Find a *counterexample* to the conjecture:

    $A \cup (B \cap C) = (A \cup B) \cap C$

    Hint: Find sets A, B, and C that make the statement false.

30. Use the sets $A = \{1, 2, 3\}$, $B = \{2, 3, 7\}$, and $C = \{3, 7, 9\}$ to give some evidence for a *distributive property* of sets:

    $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

    Does this example constitute a proof of a distributive property of sets?

❑ *FINDING ALL THE SUBSETS OF A SET*

EXAMPLE 2: **List all the subsets of {1, 2, 3}.**

Solution: Recall the Important Properties section a few pages back. There, we learned that the null set, ∅, is a subset of any set; thus ∅ is one of the subsets of {1, 2, 3}. Also, we discussed the fact that any set is a subset of itself — thus, {1, 2, 3} is one of the subsets of {1, 2, 3}. That's two subsets so far — essentially the "smallest" and the "largest" subsets.

Next, we'll list the subsets containing exactly one element:

$\{1\} \quad \{2\} \quad \{3\}$

And finally, we'll list the subsets containing exactly two elements:

$\{1, 2\} \quad \{1, 3\} \quad \{2, 3\}$

Remembering the null set and the entire set, we obtain a total of eight subsets:

$$\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}$$

EXAMPLE 3: **List all the subsets of {*a*, *b*, *c*, *d*}.**

Solution: We'll organize the subsets in order of cardinality (the number of elements in the subset):

0 elements:	$\varnothing$
1 element:	$\{a\} \quad \{b\} \quad \{c\} \quad \{d\}$
2 elements:	$\{a, b\} \quad \{a, c\} \quad \{a, d\} \quad \{b, c\} \quad \{b, d\} \quad \{c, d\}$
3 elements:	$\{a, b, c\} \quad \{a, b, d\} \quad \{a, c, d\} \quad \{b, c, d\}$
4 elements:	$\{a, b, c, d\}$

Count 'em all up — there should be **16 subsets** altogether.

Homework

31. a. List all the subsets of $\{a, b\}$.

    b. List all the subsets of $\{x\}$.

    c. List all the subsets of $\emptyset$.

32. <u>How many</u> subsets does the set $\{a, b, c, d, e, f, g, h, i, j\}$ have?

33. Suppose a set has cardinality n. How many subsets does the set have?

34. How many subsets does the set $\{\{1\}, \{1, 2\}, \{7, 9, 10\}\}$ have?

35. A set X has 128 subsets. Find the cardinality of X.

Review Problems

36. What's the distinction between a set and a non-set?

37. a. T/F: $\{5\} = 5$

    b. $A \cup A =$ _____

    c. $A \cap A =$ _____

38. We know that $\mathbb{N}$ has cardinality ∞. What is the cardinality of $\{\mathbb{N}\}$?

39. T/F: $\{1, 3\} \in \{0, 1, 2, 3, 4, 5\}$

40. Explicitly list the elements of the set $A = \{x \in \mathbb{N} \mid x \text{ is even } \textbf{and } x < 7\}$.

41. T/F: If $A \subseteq B$ and $C \subseteq B$, then $A \cup C = B$.

42. For any set A,

$$A \cap \varnothing = \_\_\_\_\_ \quad \text{and } A \cup \varnothing = \_\_\_\_\_.$$

43. If $A = \{4, 5, 9\}$, find a set B such that $A \cap B = \{5, 9\}$.

44. How many subsets does the set $\{5, 12, 3, a, 7, x, \frac{3}{8}\}$ have?

45. a. How many subsets does $\varnothing$ have?

    b. How many subsets does $\{\varnothing\}$ have?

46. Find the cardinality of a set which has 512 subsets.

47. T/F: $\{5, 6\} = \{6, 5\}$

48. The set C has cardinality 99. What is the cardinality of $\{C\}$?

49. T/F: $\{1, 3\} \subseteq \{0, 1, 2, 3, 4, 5\}$

50. Explicitly list the elements of the set $A = \{x \in \mathbb{N} \mid x \text{ is odd and } x < 10\}$.

51. T/F: If $A \subseteq B$ and $B \subseteq C$, then $C \subseteq A$.

52. For any set A,

$$A \cap A = \_\_\_\_\_ \quad \text{and } A \cup A = \_\_\_\_\_.$$

53. If $A = \{4, 5, 9\}$, find a set B such that $A \cup B = \{1, 4, 5, 9\}$.

54. How many subsets does the set $\{5, 12, 3, a, 7, y, z, w\}$ have?

55. a. How many subsets does $\varnothing$ have?
    b. How many subsets does $\{\varnothing, \{\varnothing\}\}$ have?

56. Find the cardinality of a set which has 1024 subsets.

57. T/F: For any sets A and B, $A \cup B = A \cap B$. Prove your answer.

58. True/False:

a. $\{1\} \in \{ 0, 2, \{1\}, \pi \}$

b. If Y is any set, $Y \subseteq Y$.

c. If Z is any set, $\varnothing \in Z$.

d. $\varnothing \subseteq \varnothing$.

e. $\{a, b, c\} \cup \{b, c, d\} = \{c\}$.

f. $\{1, 2\} \cap \{3, 9\} = \varnothing$.

g. If A is any set, $A \cup \varnothing = \varnothing$.

h. $\{1, 2\}$ and $\{3, \pi\}$ are disjoint.

i. If B is any set, B and $\varnothing$ are disjoint.

j. A set has 5 elements. It follows that the set has 25 subsets.

k. A set has 2048 subsets. It follows that the set has 11 elements.

l. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

m. If $A \subseteq B$, then $A \cup B = B$.

n. If $A \subseteq B$, then $A \cap B = A$.

❑ *TO ∞ AND BEYOND*

Let A be any set. We will denote the ***cardinality*** of A (the number of elements in A) by the notation $\boldsymbol{n}$**(A)**. We also define $\boldsymbol{\mathcal{P}}$**(A)**, called the ***power set*** of A, to be the set of all subsets of A.

For example, let A = {1, 5}. Then

$$n(A) = 2 \text{ and } \mathcal{P}(A) = \{ \varnothing, \{1\}, \{5\}, \{1, 5\} \}$$

Can you see that $n(\mathcal{P}(A)) = 4$?

A. State the definition of $\mathcal{P}$(A) (fill in the blank):

$$\mathcal{P}(A) = \{ X \mid \underline{\qquad\qquad} \}$$

B. If A is a set such that n(A) = 12, calculate $n(\mathcal{P}$(A)).

C. Suppose B is a set such that $n(\mathcal{P}$(B)) = 256. Calculate n(B).

Solutions

1. Yes; well-defined.

2. No; exactly what is a high GPA?

3. $\in$, $\notin$

4. $\in$, $\subseteq$

5. a. T b. F

6. F

7. 8 is an element of the first set, but it's not an element of the second set.

8. F — one set has 1 element in it, while the other set has 0 elements in it.

9. The first

10. Every natural number is an integer, so $\mathbb{N} \subseteq \mathbb{Z}$. But $-3 \in \mathbb{Z}$, yet $-3 \notin \mathbb{N}$; thus $\mathbb{Z} \not\subseteq \mathbb{N}$.

11. A = B

12. T

13. {1, 2, 3, 4, 5}

14. {5, 6, 7, 8, 9}

15. $\{x \in \mathbb{N} \mid x \text{ is even and } x \leq 10\}$

16. $\{x \in \mathbb{N} \mid x \text{ is a prime number} \leq 13\}$

17. T — duplicates can be discarded, and the order doesn't matter.

18. F

19. T

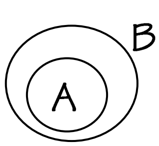
20. T — the null set is a subset of <u>every</u> set, including itself.

21. 0; 11

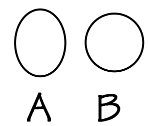
22. 1

23. ∞; ∞; What do you think?

24.



25. Draw A inside B and B inside C. Is it clear that A is inside C, and therefore that $A \subseteq C$?

26. $\{a, b, c, d, w, z, m\}$ **27**. {5} **28**. 

29. Let me know what sets you came up with.

30. Work out each side by doing what's in parentheses first. An example does not prove anything.

31. a. $\{a, b\}, \{a\}, \{b\}, \varnothing$ b. $\{x\}, \varnothing$ c. $\varnothing$

32. 1024 **33**. What do you think? **34**. 8 **35**. 7

36. A set is well-defined; it's always possible to tell whether something is in the set or not.

37. a. False b. A c. A **38**. 1 **39**. False **40**. A = {2, 4, 6}

41. False **42**. $\varnothing$; A **43**. B = {5, 7, 9}, for example

44. 128 **45**. a. 1 b. 2 **46**. 9

47. a. T **48**. 1 **49**. T

50. A = { 1, 3, 5, 7, 9} **51**. F **52**. A; A **53**. B = {1}

54. 256 **55**. a. 1 b. 4 **56**. 10

57. False; for example, let A = {1, 2} and B = {3}. Then $A \cup B = \{1, 2, 3\}$, while $A \cap B = \varnothing$.

58. a. T b. T c. F d. T e. F f. T g. F h. T
i. T j. F k. T l. T m. T n. T

"Our progress as a nation can be no swifter than our progress in education."

– John Fitzgerald Kennedy